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Generation of magnetic fields by convection

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The nonlinear hydromagnetic dynamo problem is investigated for the case of convection in a layer of an electrically conducting fluid heated from below. It is shown that two-dimensional convection rolls in conjunction with a longitudinal mean flow are capable of amplifying a magnetic field in the form of a wave propagating in the longitudinal direction. The action of the Lorentz forces causes a reduction of the amplitude of convection with the consequence that the energy of the magnetic field cannot grow beyond an equilibrium value which is determined as a function of the parameters of the problem. The analysis is based on an expansion in powers of the longitudinal wavenumber β of the magnetic field and applies in the case of large values of the magnetic Prandtl number.

1. Introduction

The problem of the generation of magnetic fields by motions in a conducting fluid is known as the dynamo problem. An increasing number of theoretical investigations of this problem at different levels of mathematical complexity have appeared in recent years. Most of the work has been focused on the kinematic dynamo problem, which is concerned with the conditions under which growing solutions of the linear dynamo equation

$$(\partial/\partial t - \lambda\nabla^2)\,\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) \qquad (1.1)$$

for the magnetic field $\mathbf{B}$ exist. Arbitrary solenoidal vector fields are admitted as velocity fields $\mathbf{u}$ in (1.1). A recent review of the work on this problem has been given by P. H. Roberts (1971).

The shortcomings of the linear homogeneous problem (1.1) are that the amplitude of the magnetic field is not determined and that the question of the physical realizability of the velocity field remains unanswered. The complete dynamo problem consists of the equations of motion and equation (1.1), which are coupled by the action of the Lorentz forces. The generation of magnetic fields manifests itself initially as the instability of the solution of a hydrodynamical problem without Lorentz forces. As the magnetic field grows the Lorentz forces become important and modify the velocity according to the Lenz rule in such a way that the amplification of the magnetic field is reduced. Asymptotically this action leads to an equilibrium amplitude for the magnetic field, at least in the sense of a time average.

In view of the complexities of the kinematic dynamo problem the mathematical difficulties of the nonlinear hydromagnetic dynamo problem appear to be almost prohibitive. Nevertheless, considerable progress in this direction has been made recently by Moffatt (1970, 1972), who has solved a statistical version of the problem which is based on the solution of the kinematic turbulent dynamo problem by Steenbeck, Krause & Rädler (1966). Although the equations of motion have been taken into account by Moffatt, only the action of random forces has been considered.

The present paper describes an attempt to solve the nonlinear hydromagnetic dynamo problem without statistical assumptions in a physically realistic situation. We have chosen the case of convection in a layer heated from below, which can be regarded as a representative of the convection flows which generate magnetic fields on the sun and, perhaps, in the earth. Since the most simple solution for convection in the form of rolls cannot generate magnetic fields, we shall consider the problem with an additional plane parallel shear flow along the axis of the convection rolls. Although this velocity field depends only on two co-ordinates, it is complex enough to yield growing solutions of (1.1).

The analysis of the problem starts in §2 with the consideration of the Boussinesq equations for convection without Lorentz forces. Small amplitude solutions for the convection rolls and the additional mean flow are described in this section. The linear dynamo problem based on (1.1) is formulated in §3. Since the velocity field is two-dimensional it can be assumed that the magnetic field has the form of a plane wave with respect to the third dimension. The corresponding wavenumber β is regarded as a small parameter and an expansion of the equations in powers of β is used in §4 to obtain the solution. The main result, found in the order β^2 equations, is that positive growth rates of the magnetic field become possible when the magnetic Reynolds number based on the amplitude of convection exceeds a critical value of order 10. The major effect of the Lorentz force is the reduction of the amplitude of convection. This property permits a relatively simple calculation of the equilibrium amplitude of the magnetic field which is given in §5. The paper closes with a general discussion in §6.

2. The equations of motion without Lorentz forces

We consider a horizontal fluid layer of height d with a temperature T_2 prescribed at the upper boundary. For the non-dimensional description of the problem we introduce d, d^2/ν and $(T_2-T_1)P$ as scales for length, time and temperature, respectively, where ν is the kinematic viscosity and P is the Prandtl number. The corresponding scale for the velocity field is ν/d. We shall use a Cartesian system of co-ordinates with the origin at the lower boundary and the x axis in the vertical direction. The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ will be used as unit vectors in the direction of the x, y and z co-ordinates, respectively. We assume that in addition to the convection generated by the gravitational instability of the static fluid layer a mean flow $W(x)$ in the direction of the z axis is present. Since we are assuming stress-free boundaries

$$\mathbf{i}.\mathbf{v} = 0, \quad \mathbf{i}.\nabla\mathbf{i}\times\mathbf{v} = 0 \quad \text{at} \quad x = 0, 1, \tag{2.1}$$

the mean flow cannot be generated by a pressure gradient. In order to generate a mean flow with a symmetric profile of the form

$$W(x) = W_0 \cos 2\pi x \tag{2.2}$$

we assume that a temperature distribution of the form

$$\tau = \tau_0 z \sin 2\pi x \tag{2.3}$$

is superimposed on the linear temperature distribution produced by the temperatures at the boundaries. We shall not go into the details of the physical mechanism which causes the deviation τ from the linear profile A distribution of heat sources and sinks or small variations of the conductivity would be suitable candidates for this mechanism. To ensure that the additional distribution (2.3) can be regarded as a small perturbation, we shall assume that

$$\tau_0 L \ll P^{-1}, \tag{2.4}$$

where L is a typical length scale of the system in the z direction. L will be regarded as a large parameter such that the limit $L \to \infty$ can be assumed as long as condition (2.4) is satisfied.

The Boussinesq equations of motion without Lorentz forces for the velocity $\mathbf{v}$ and the heat equation for the deviation θ of the temperature from the basic temperature distribution $T = P^{-1}\{T_2/(T_2-T_1)-x\}+\tau$ are

$$\nabla^2\mathbf{v}+\mathbf{i}R(\theta+\tau)-\nabla\pi = \mathbf{v}.\nabla\mathbf{v}+\partial\mathbf{v}/\partial t, \tag{2.5a}$$

$$\nabla.\mathbf{v} = 0, \tag{2.5b}$$

$$\nabla^2\theta+\mathbf{i}.\mathbf{v} = P(\mathbf{v}.\nabla\theta+\partial\theta/\partial t). \tag{2.5c}$$

The Rayleigh number R and the Prandtl number P are defined by

$$R \equiv \gamma g d^3(T_2-T_1)/\kappa\nu, \quad P \equiv \nu/\kappa,$$

where γ is the coefficient of thermal expansion, g is the acceleration of gravity and κ is the thermal diffusivity. Using assumption (2.4) we have neglected the term $\mathbf{v}.\nabla\tau$ in the heat equation. The exact conditions for which the equations are valid will be discussed below.

It is known that the shearing action of the mean flow exerts, in general, a stabilizing influence by increasing the critical value R_c of the Rayleigh number for all convection flows with the exception of two-dimensional rolls aligned with the direction of the mean flow (e.g. Gage & Reid 1968). We shall for this reason restrict attention to the latter case and assume that the velocity field $\mathbf{v}$ is z-independent.

The equation of continuity (2.5b) can be eliminated by using the following representation for $\mathbf{v}$:

$$\mathbf{v} = \nabla\times\mathbf{k}\psi(x,y)+\mathbf{k}w(x,y). \tag{2.6}$$

Assuming the stationary case we obtain from (2.5) by taking the y and the z component of the curl of the equation of motion

$$\nabla_2^2\partial_x w - R\partial_z\tau = \partial_x(\boldsymbol{\epsilon}\psi.\nabla w), \tag{2.7a}$$

$$\nabla_2^4\psi + R\partial_y\theta = \boldsymbol{\epsilon}.\{\boldsymbol{\epsilon}\psi.\nabla\boldsymbol{\epsilon}\psi\}, \tag{2.7b}$$

$$\nabla_2^2\theta+\partial_y\psi = P\{\boldsymbol{\epsilon}\psi.\nabla\theta\}, \tag{2.7c}$$

where the two-dimensional Nabla operator ∇_2 has the components $(\partial_x, \partial_y, 0)$ and the operator $\boldsymbol{\epsilon}$ is defined by

$$\boldsymbol{\epsilon}\psi \equiv \nabla \times \mathbf{k}\psi. \tag{2.8}$$

Equations (2.7) confirm our expectation that in the limit (2.4) a z-independent solution is possible. For the case when the amplitude of ψ is small compared with unity equation (2.7 *a*) is solved by a mean flow of the form (2.2), $w = W(x)$, with

$$W_0 = R\tau_0/8\pi^3. \tag{2.9}$$

The fact that equations (2.7 *b*, *c*) are independent of w not only establishes the property that the critical Rayleigh number for longitudinal convection rolls is independent of the mean flow, but also shows that the finite amplitude properties and the heat transport in particular remain unaffected by the presence of the shear.

Solutions of (2.7 *b*, *c*) are well known from earlier work on convection (e.g. Schlüter, Lortz & Busse 1965). The solution for stress-free boundaries and small amplitudes A can be written in the form

$$\psi_0 = A\{\sin\alpha y \sin\pi x + O(A^2P^2)\}, \tag{2.10a}$$

$$R = R_\alpha + \tfrac{1}{8}P^2A^2(\pi^2+\alpha^2)^2 + \ldots, \tag{2.10b}$$

where $R_\alpha = (\pi^2+\alpha^2)^3/\alpha^2$ is the critical value of the Rayleigh number for disturbances with the horizontal wavenumber α. Unless special initial conditions are used convection with the wavenumber

$$\alpha_c = \pi/\sqrt{2} \tag{2.11}$$

corresponding to the minimum of R_α will be realized in physical situations. In the following we shall neglect the higher order terms of magnitude A^2P^2 except in expression (2.10 *b*), which provides the relation between the convection amplitude A and the Rayleigh number. If we do not assume that $L\tau_0 \ll A$ a term of the order $(PL\tau_0)^2$ has to be added on the right-hand side of (2.10 *b*) if we assume for simplicity that the geometry of the system is symmetric with respect to $z = 0$. However, since a term of the order $P^2\tau_0 LA$ does not exist for reasons of symmetry the relation between the Rayleigh number and the amplitude A remains unchanged. The term of order $(P\tau_0 L)^2$ just causes a slight change in the value of the constant R_α. As long as condition (2.4) is satisfied the analysis holds even for the case

$$W_0 \approx A, \tag{2.12}$$

which requires $L\tau_0 \gg A$ in the limit $L \gg 1$. The variation of the amplitude of convection as a function of z is of order $L\tau_0 PA$ and can be neglected since it is small compared with the mean amplitude A because of condition (2.4). Finally, it is worth mentioning without going into details that the velocity field (2.6) with (2.10 *a*) and (2.11) is hydrodynamically stable for sufficiently small amplitudes since convection rolls are stable in contrast to three-dimensional convection flows (Schlüter *et al.* 1965).

3. The dynamo equation

In order to introduce a non-dimensional version of the dynamo equation (1.1) we shall measure the magnetic field in terms of the non-dimensional Alfvén velocity

$$\mathbf{B} = (\rho\mu)^{\frac{1}{2}}\nu d^{-1}\mathbf{H}, \tag{3.1}$$

where $\mathbf{H}$ denotes the dimensionless magnetic field, ρ is the density and μ is the magnetic permeability. For the analysis of the dynamo equation it is convenient to use d^2/λ as the time scale rather than d^2/ν as we have done in the previous section. This means that we have to multiply the velocity field derived above by the factor $P_m = \nu/\lambda$ in order to use it in the dimensionless dynamo equation. P_m is the magnetic Prandtl number and describes the ratio of the viscous diffusivity to the magnetic diffusivity. In order to simplify the notation we shall indicate the multiplication by P_m of variables defined in the previous section by a star, for example

$$\psi_0^* \equiv P_m\psi_0.$$

Solutions of (1.1) are always solenoidal when their initial values have this property. Even though the condition $\nabla.\mathbf{H} = 0$ does not have to be taken into account it is convenient to use a general representation for $\mathbf{H}$ in terms of two scalar potentials as in the case of the velocity field. Before we introduce this representation we note that it can be assumed without losing generality that the solution of the dynamo equation (1.1) has an exponential dependence on z as well as on the time when the velocity field $\mathbf{v}$ is stationary and z-independent as we have assumed. Accordingly,we assume as a representation for the magnetic field $\mathbf{H}$

$$\mathbf{H} = H_A[(-\mathbf{j}+\boldsymbol{\epsilon}g)\exp\{\sigma t+i\beta z\}+\nabla\times(\nabla\times\mathbf{k}h\exp\{\sigma t+i\beta z\})], \tag{3.2}$$

where the potentials g and h are functions of x and y only.

We have separated the x, y average of the magnetic field from the fluctuating part, which is described by the functions g and h. The boundary conditions discussed below require that the mean magnetic field lies in the y direction unless β vanishes. The latter case, however, is not of interest to us since it corresponds to a two-dimensional field, which cannot be created by fluid motions according to Cowling's theorem. Using the velocity field

$$\mathbf{v} = \boldsymbol{\epsilon}\psi_0+\mathbf{k}W$$

derived above we find by taking the average of the y component of (1.1)

$$\sigma+\beta^2 = i\beta(\overline{W^*(i\beta\partial_y h-\partial_x g)}-\overline{\partial_x\psi_0^*\nabla_2^2 h}). \tag{3.3a}$$

The average over the x and y co-ordinates has been indicated by a bar. By taking the z component of the curl of (1.1) we find

$$(\sigma-\nabla_2^2+\beta^2)\nabla_2^2 g = -\nabla_2^2(\boldsymbol{\epsilon}\psi_0^*.\nabla g)-\nabla_2^2\partial_y\psi_0^*-i\beta\{\boldsymbol{\epsilon}.[W^*(i\beta\nabla_2 h+\boldsymbol{\epsilon}g)] + \partial_x W^*\nabla_2^2(\nabla_2 h.\nabla_2\psi_0^*)+\boldsymbol{\epsilon}.(\boldsymbol{\epsilon}\psi_0^*\nabla_2^2 h)\}. \tag{3.3b}$$

The z component of (1.1) yields as the equation for h

$$(\sigma-\nabla_2^2+\beta^2)\nabla_2^2 h = -\nabla_2.[W^*(i\beta\nabla_2 h+\boldsymbol{\epsilon}g)]+\partial_y W^*-\boldsymbol{\epsilon}\psi_0^*.\nabla\nabla_2^2 h. \tag{3.3c}$$

Equations (3.3) hold for general functions W^* and ψ_0^*. For this reason the term $\partial_y W^*$ has been included in the last equation although it vanishes for the mean flow (2.2) which will be used in the analysis of the equations. We shall assume that the boundaries are perfect electrical conductors, with the consequence that the normal component of the magnetic field and the tangential component of the current density vanish at the boundaries. These conditions require

$$\partial_y g = \partial^2_{xx} g = \partial_x h = \partial^3_{xxx} h = 0 \quad \text{at} \quad x = 0, 1. \tag{3.4}$$

It is known from the work of Elsasser (1946) and Bullard & Gellman (1954) that a toroidal velocity field cannot generate magnetic fields. A velocity field confined to parallel planes as the field described by ψ_0 can be regarded as a limiting case of a toroidal velocity field. It is readily seen from equations (3.3) that no generation of magnetic fields can take place for $W = 0$. Equation (3.3c) for $\nabla_2^2 h$ becomes identical to the heat equation in this case and requires that $\nabla_2^2 h$ decays since no source term is available. After $\nabla_2^2 h$ and hence h have vanished the same argument can be applied for g since (3.3b) becomes the heat equation after $h = W = 0$ has been used and the operator ∇_2^2 has been removed by integration.

The fact that convection in the form of rolls cannot generate magnetic fields has prompted us to add the mean flow $W(x)\,\mathbf{k}$ to the problem. A number of possibilities for two-dimensional velocity fields which lead to generation of magnetic fields has been discussed recently by G. O. Roberts (1972). Yet, there are few examples which correspond to physically realizable situations. An example which comes to mind are convection rolls in a layer heated from below and rotating about a vertical axis. An inspection shows, however, that this form of convection still can be considered as a toroidal velocity field even though a component of velocity field in the direction of the axis of the rolls does exist. Since convection with a symmetric mean flow appears to be the next simple possibility it has been chosen for the purpose of the present analysis. Hydromagnetic dynamos with more complex convection flows can become accessible to a relatively simple analysis if the velocity field possesses scales of different order. A promising approach in this direction has been recently proposed by Childress & Soward (1972).

4. Solution of the linear dynamo problem

We shall solve equations (3.3) for the case of small but finite wavenumbers β. For this purpose we expand the dependent variables in series of powers of β:

$$\left.\begin{aligned} \sigma &= \sigma^{(0)} + i\beta\sigma^{(1)} + \beta^2\sigma^{(2)} + \ldots, \\ g &= g^{(0)} + i\beta g^{(1)} + \beta^2 g^{(2)} + \ldots, \\ h &= h^{(0)} + i\beta h^{(1)} + \beta^2 h^{(2)} + \ldots . \end{aligned}\right\} \tag{4.1}$$

We have anticipated from the symmetry of (3.3) that even orders in β are associated with real quantities while odd orders lead to purely imaginary expressions. After introducing the expansion (4.1) in (3.3) we find from the zeroth-order form of (3.3a)

$$\sigma^{(0)} = 0. \tag{4.2}$$

This result is not unexpected since the dissipation of the mean field depends on its z dependence. Without a mechanism of generation the decay rate of the magnetic field will be proportional to β^2. A non-trivial result is found from the zeroth-order forms of (3.3*b*) and (3.3*c*):

$$(\nabla^2 - \boldsymbol{\epsilon}\psi_0^* . \nabla)\, g^{(0)} = \partial_y \psi_0^*, \tag{4.3a}$$

$$(\nabla^2 - \boldsymbol{\epsilon}\psi_0^* . \nabla)\, \nabla^2 h^{(0)} = \boldsymbol{\epsilon} g^{(0)} . \nabla W^*. \tag{4.3b}$$

Since all variables depend only on x and y we are using for simplicity ∇ instead of ∇_2 here and in the following. Before solving equations (4.3) we consider the equations of higher order. The equations of first order do not yet answer the question whether generation of magnetic fields can occur. A non-vanishing value of $\sigma^{(1)}$ indicates that the magnetic field is propagating in the direction of the z axis:

$$\sigma^{(1)} = -\overline{W^* \partial_x g^{(0)}} - \overline{\partial_x \psi_0^* \nabla^2 h^{(0)}}. \tag{4.4}$$

Since the generation mechanism is governed by terms of order β^2 the functions $g^{(1)}$ and $h^{(1)}$ have to be determined:

$$\nabla^2(\nabla^2 - \boldsymbol{\epsilon}\psi_0^* . \nabla)\, g^{(1)} = -\nabla^2(\nabla h^{(0)} . \nabla\psi_0^*) + \boldsymbol{\epsilon} . (\boldsymbol{\epsilon}\psi_0^* \nabla^2 h^{(0)}) + \sigma^{(1)} \nabla^2 g(0)$$
$$+ \boldsymbol{\epsilon}(W^* . \boldsymbol{\epsilon} g^{(0)}) + \partial_x W^*, \tag{4.5a}$$

$$(\nabla^2 - \boldsymbol{\epsilon}\psi_0^* . \nabla)\, \nabla^2 h^{(1)} = \boldsymbol{\epsilon} g^{(1)} . \nabla W^* + \nabla . (\nabla h^{(0)} W^*) + \sigma^{(1)} \nabla^2 h^{(0)}. \tag{4.5b}$$

After the solution of these equations has been obtained the real part of the growth rate to lowest order can be determined from

$$\sigma^{(2)} = -1 - \overline{W^* \partial_y h^{(0)}} + \overline{W^* \partial_x g^{(1)}} + \overline{\partial_x \psi_0^* \nabla^2 h^{(1)}}. \tag{4.6}$$

It is the main goal of the following analysis to evaluate this expression.

In order to solve the linear partial differential equations (4.3) and (4.5) we use a Fourier representation for the variables. The trigonometric functions $\sin n\pi x$ and $\cos n\pi x$ are especially appropriate for this purpose since they satisfy the boundary conditions (3.4) for g and h, respectively. The y dependence of ψ_0 suggests the orthogonal systems $\sin m\alpha y$ and $\cos m\alpha y$ for the representation of the y dependence. After considering the symmetry of the equations we find as representations for $g^{(0)}$ and $h^{(0)}$

$$g^{(0)} = \sum_{n,m} g_{nm}^{(0)} \sin n\pi x \cos m\alpha y, \tag{4.7a}$$

$$h^{(0)} = W_0^* \sum_{n,m} h_{nm}^{(0)} \cos n\pi x \sin m\alpha y. \tag{4.7b}$$

Since the dynamo equation is of the Mathieu type with respect to the y dependence the general representation for $g^{(0)}$ and $h^{(0)}$ has to include a factor $\exp\{ily\}$ on the right-hand side of (4.7*a*, *b*). From the experience with similar problems in the theory of convection (Busse 1967) we expect that the largest growth rates correspond to the case $l = 0$. Because we are restricting attention to the case $l = 0$ and to the limit $\beta \ll 1$ we cannot exclude the possibility that other magnetic fields can be generated at magnetic Reynolds numbers lower than those obtained in the present analysis. We have extracted the factor W^* in the representation (4.7*b*) of $h^{(0)}$. From (4.3) and (4.5) it can be seen that in general $g^{(n)}$ is proportional to W_0^{*n} while $h^{(n)}$ is proportional to W_0^{*n+1}. We use this fact by extracting the factor W_0^* and W_0^{*2} in the representations of $g^{(1)}$ and $h^{(1)}$, respectively, which are otherwise analogous to (4.7). Because of the symmetry of the equations only terms with $n+m$ even have to be considered in the summation. By multiplying (4.3a) and (4.5a) by $4\sin n\pi x \cos m\alpha y\,(1-\frac{1}{2}\delta_{0m})$ and (4.3b) and (4.5b) by $4\cos n\pi x$ $\sin m\alpha y\,(1-\frac{1}{2}\delta_{n0})$ and averaging the result we obtain the following system of algebraic equations:

$$A_{nm\nu\mu}g^{(0)}_{\mu\nu} = A^*\alpha\delta_{1n}\delta_{1m}, \tag{4.8a}$$

$$B_{nm\nu\mu}h^{(0)}_{\nu\mu} = E_{nm\nu\mu}g^{(0)}_{\nu\mu}, \tag{4.8b}$$

$$A_{nm\nu\mu}g^{(1)}_{\nu\mu} = \sigma^{(1)}g^{(0)}_{nm} + A^*H_{nm\nu\mu}g^{(0)}_{\nu\mu} + F_{nm\nu\mu}g^{(0)}_{\nu\mu} + (2\pi)^{-1}\delta_{2n}\delta_{0m}, \tag{4.8c}$$

$$B_{nm\nu\mu}h^{(1)}_{\nu\mu} = \sigma^{(1)}h^{(0)}_{nm} + E_{nm\nu\mu}g^{(1)}_{\nu\mu} + K_{nm\nu\mu}h^{(0)}_{\nu\mu}, \tag{4.8d}$$

where the summation convention applies for subscripts occurring twice in the same term. Explicit expressions for the matrices with four subscripts are given in the appendix. The relations (4.4) and (4.6) for $\sigma^{(1)}$ and $\sigma^{(2)}$ can be evaluated very simply once the solutions of (4.8a, b) and (4.8c, d) have been obtained:

$$\sigma^{(1)} = W_0^* f(A^*) \quad \text{with} \quad f(A^*) \equiv \tfrac{1}{4}\pi A^*(\pi^2+\alpha^2)\,h^{(0)}_{11} - \pi g^{(0)}_{20}, \tag{4.9}$$

$$\sigma^{(2)} = -1 + W_0^{*2} b(A^*) \quad \text{with} \quad b(A^*) \equiv -\tfrac{1}{4}\pi A^*(\pi^2+\alpha^2)\,h^{(1)}_{11} + \pi g^{(1)}_{20}. \tag{4.10}$$

In order to solve equations (4.8) numerically the infinite matrices have to be truncated. We implement the truncation by neglecting elements and equations with subscripts n, m wherever $n+m > N$. The truncation parameter N is allowed to vary and it is anticipated that the solution and in particular the values of $\sigma^{(1)}$ and $\sigma^{(2)}$ will not change at sufficiently large values of N if N is replaced by $N+2$. We find that the convergence is indeed excellent for moderate values of AP_m. For the values of A^* used in plotting the figures the values of $\sigma^{(1)}$ and $\sigma^{(2)}$ change only in the fifth decimal place when $N = 8$ is replaced by $N = 10$.

It would consume too much space to present the numerical results of the computations of $g^{(0)}$, $g^{(1)}$, $h^{(0)}$ and $h^{(1)}$. We shall restrict attention for this reason to the results for the expressions (4.9) and (4.10). Some further results will be presented later when we discuss the action of the Lorentz forces. Of primary interest is the function $b(A^*)$, which is shown in figure 1. It becomes positive when A^* exceeds the critical value

$$A_c^* = 2{\cdot}224 \quad \text{for} \quad \alpha = \pi/\sqrt{2}. \tag{4.11}$$

Since A^* occurs in the problem mostly in the combination αA^* the critical value is slightly smaller for larger values of α even though the dissipative action increases with α. Depending on the magnitude of W_0^* positive values of $\sigma^{(2)}$ and thus growth of the magnetic field become possible for values of A^* in the supercritical region. By assuming P_m sufficiently large arbitrarily high values of A^* and W_0^* can be obtained in spite of the assumption $A, W_0 \ll 1$ made in the hydrodynamic analysis. However, the growth rate $\sigma^{(2)}$ does not increase very much if P_m is increased beyond a certain range for given values of A and W_0 since the function $b(A^*)$ decreases, being nearly proportional to A^{*-2}. Computations in the range between $A^* = 30$ and $A^* = 150$ were carried out with truncation

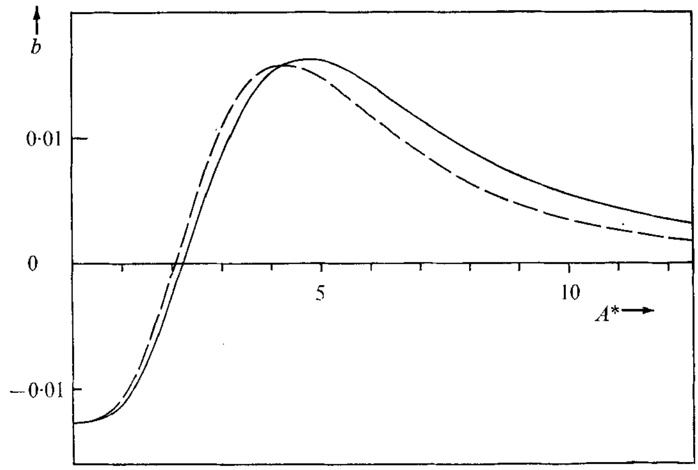

FIGURE 1. The function $b(A^*)$ which describes the dependence of the growth rate $\sigma^{(2)}$ on the amplitude of convection. The solid curve and the dashed curve correspond to $\alpha = \pi/\sqrt{2}$ and $\alpha = \pi$, respectively.

	N					
A^*	12	14	16	18	20	22
30	6·6683	6·6730	—	—	—	—
40	3·9481	3·9519	3·9533	—	—	—
60	1·8846	1·8776	1·8769	—	—	—
80	1·1271	1·1095	1·1046	—	—	—
100	—	—	0·7333	0·7301	—	—
120	—	—	0·5264	0·5219	0·5198	0·5186
150	—	—	0·3533	—	—	0·3429

TABLE 1. Values of $b^*(A) \times 10^4$ at large values of A^* for $\alpha = \pi/\sqrt{2}$

parameters up to $N = 22$ and are shown in part in table 1. The results show that $b(A^*)$ can be represented in this range within fractions of 1 % by

$$b(A^*) = 0{\cdot}3658 A^{*-1{\cdot}85}. \tag{4.12}$$

This decay is caused by the fact that the magnetic field is expelled from most of the convection layer and compressed into thin boundary layers where the generation process is no longer effective. The tendency towards this state is evident from figure 2, which shows the toroidal part of the magnetic field at the moderate value $A^* = 5$. We conclude from this consideration that the optimal conditions for amplification of the magnetic field are attained at moderate values, of order 10, of the magnetic Reynolds number $R_m = \alpha A^*$.

The non-vanishing expression for $\sigma^{(1)}$ shows that the magnetic field has the form of a wave travelling in the direction of the mean flow at the boundary of the layer. The phase velocity, which is given in first approximation by $\sigma^{(1)}$, increases quadratically with A^* for small values of A^* and reaches a constant value for large values of A^* as shown in figure 3.

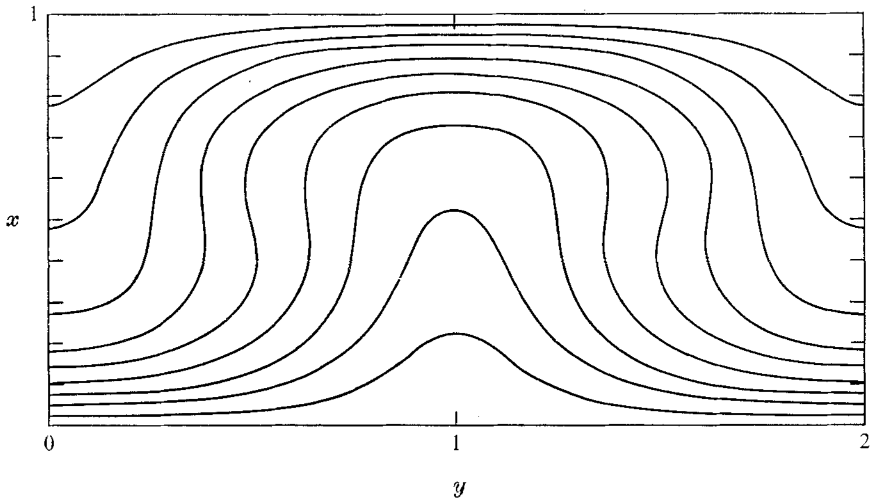

FIGURE 2. The toroidal part of the magnetic field described by $\mathbf{H} = -\mathbf{j} + \epsilon g^{(0)}$ in the presence of convection with the amplitude $A^* = 5$ and the wavenumber $\alpha = \pi$. The field-lines correspond to constant values of the function $x + g^{(0)}(x, y)$ and are shown for increments of 0·1.

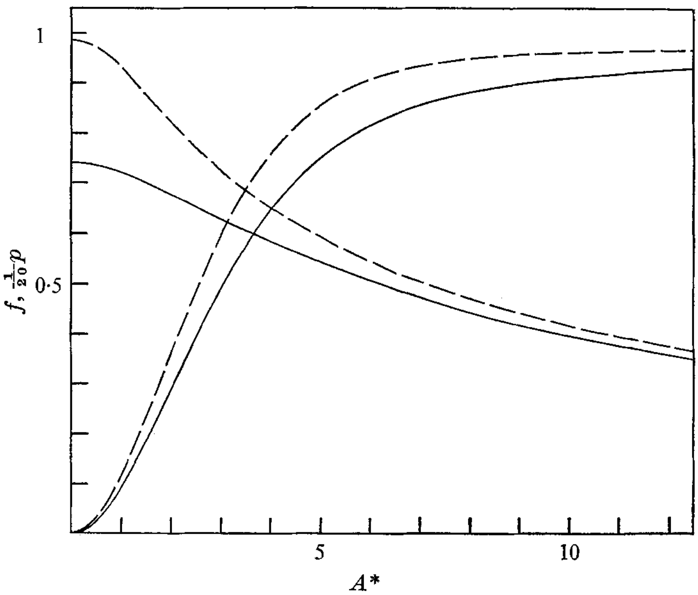

FIGURE 3. The increasing function $f(A^*)$ describes the dependence of the frequency $\sigma^{(1)}$ on the amplitude of convection. The decreasing function $p(A^*)$ describes the work done by the Lorentz forces divided by A^2. The solid curve and the dashed curve correspond to $\alpha = \pi/\sqrt{2}$ and $\alpha = \pi$, respectively.

Some insight into the generation mechanism can be gained from considering the equations with the truncation parameter $N = 2$. If we neglect terms of order A^{*3} we find that

$$g_{11}^{(0)} = \frac{-\alpha A^*}{\pi^2+\alpha^2}+\ldots, \quad g_{20}^{(0)} = \frac{\alpha^2 A^{*2}}{8\pi(\pi^2+\alpha^2)}+\ldots, \quad h_{20}^{(0)} = \frac{-\alpha^2\pi A^*}{(\pi^2+\alpha^2)^3}+\ldots,$$

$$f(A^*) = -A^{*2}\frac{\alpha^2}{4(\pi^2+\alpha^2)}\left(\tfrac{1}{2}+\frac{\pi^2}{\pi^2+\alpha^2}\right)+\ldots, \tag{4.13}$$

$$h_{11}^{(1)} = \frac{-\alpha^2 A^*}{8\pi(\pi^2+\alpha^2)^3}+\ldots, \quad g_{20}^{(1)} = \frac{-1}{8\pi^3}+\frac{(\alpha A^*)^2}{16\pi(\pi^2+\alpha^2)}\left(\frac{1}{4\pi^2}-\frac{1}{2(\pi^2+\alpha^2)}\right)+\ldots.$$

The latter two expressions enter the relation (4.10), which defines the function $b(A^*)$,

$$b(A^*) = \frac{-1}{8\pi^2}+\frac{(\alpha A^*)^2}{64(\pi^2+\alpha^2)\,\pi^2}+\ldots, \tag{4.14}$$

which shows that the decay at low values of A^* is the result of the dissipation which is caused by the shearing action of the mean flow on the mean magnetic field. We note that we use the word 'mean' in two different ways. The mean magnetic field represents the average over the x and y co-ordinates while the mean flow stands for the average over the y and z directions.

The generation of the mean magnetic field at finite values of A^* takes place because of the stretching of the y component of the poloidal part of the magnetic field by the toroidal part of the velocity field. The stretching process causes both propagation of the magnetic wave and an amplification. At higher values of A^* and at high truncation parameters N additional positive terms appear on the right-hand side of relations (4.13) and (4.14), and are responsible for the fact that the zero of $b(A^*)$ occurs in the neighbourhood of 2 instead of 5 as could be expected from (4.14). Yet the main dynamo mechanism can still be described by the following three steps. First, a fluctuating toroidal field is created from the mean magnetic field by the action of the convection velocity field. Then a fluctuating poloidal field is created by the mean flow. Finally, the mean field is propagated and amplified by the stretching of the poloidal field by the toroidal convection velocity field.

5. The action of the Lorentz forces

As the magnetic field grows to finite amplitudes the action of the Lorentz forces has to be taken into account by adding the term

$$(\nabla\times\mathbf{H})\times\mathbf{H} \tag{5.1}$$

on the left-hand side of the equation of motion (2.5*a*). The modification of the velocity field caused by the Lorentz force will yield a relation for the amplitude of the magnetic field, which is represented by the as yet undetermined coefficient H_A. It is more convenient to use the parameter $E = \frac{1}{4}H_A^2$, which is a measure for the energy density of the magnetic field. E is exactly equal to the density of magnetic field in the limit $A^* \to 0$ if we assume that magnetic field $\mathbf{H}$ is given by the real part of expression (3.2) as we shall do in the following. The evaluation of (5.1) yields the following expression if we restrict attention to terms of lowest order in β and take the time average, which, of course, is identical with the z average:

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = E\{-(\mathbf{i}+\nabla g^{(0)})\,\nabla^2 g^{(0)} + \mathbf{k}[(\mathbf{i}+\nabla g^{(0)}) \times \nabla\nabla^2 h^{(0)}].\mathbf{k} - \tfrac{1}{2}\nabla|\nabla^2 h^{(0)}|^2\}. \quad (5.2)$$

The last term in the curly bracket does not create any fluid motion since it can be balanced by the pressure. The component of the Lorentz force in the z direction tends to decrease the amplitude of the mean flow and creates in addition a poloidal velocity field. As will be shown below, this action of the Lorentz force as well as the effects caused by components of the Lorentz forces not included in (5.2) are of minor importance compared with the action of the toroidal component of the Lorentz force.

The toroidal component of the Lorentz force leads to the additional term

$$E\boldsymbol{\epsilon}.(\mathbf{i}+\nabla g^{(0)})\,\nabla^2 g^{(0)} \quad (5.3)$$

on the right-hand side of the equation (2.7*b*) for the convection velocity field. We anticipate that E is a small parameter and assume that a solution of the modified convection equations can be obtained by a perturbation approach. Accordingly, we introduce the expansion

$$\left.\begin{aligned} \psi &= \psi_0 + E\psi_1 + \ldots,\\ R &= R_0 + ER_1 + \ldots,\\ \theta &= \theta_0 + E\theta_1 + \ldots, \end{aligned}\right\} \quad (5.4)$$

where the subscript 0 refers to the solution without magnetic field. Since A is a small parameter the nonlinear terms in the convection equation can be neglected at this point and we obtain as equations of first order in E

$$\left.\begin{aligned} &\nabla_2^4\psi_1 + R_0\,\partial_y\theta_1 = \boldsymbol{\epsilon}.\{(\mathbf{i}+\nabla g^{(0)})\,\nabla^2 g^{(0)}\} - R_1\,\partial_y\theta_0,\\ &\nabla_2^2\theta_1 + \partial_y\psi_1 = 0. \end{aligned}\right\} \quad (5.5)$$

Since the homogeneous part of equations (5.5) is self-adjoint the inhomogeneity has to be orthogonal to the solution (ψ_0, θ_0) of the homogeneous equations. Thus, we find by multiplying the first and the second equation of (5.5) by ψ_0 and $R_0\theta_0$, respectively, and averaging the result

$$\tfrac{1}{4}R_1\frac{\alpha^2 A^2}{\pi^2+\alpha^2} - \overline{\boldsymbol{\epsilon}\psi_0.(\mathbf{i}+\nabla g^{(0)})\,\nabla^2 g^{(0)}} = 0. \quad (5.6)$$

The solvability condition (5.6) determines R_1 in terms of the solution (4.7*a*):

$$P_m^{-1}R_1 = p(A^*) \equiv \frac{\pi^2+\alpha^2}{\alpha^2 A^{*2}}\sum_{n,m} d(n,m)\,g_{nm}^{(0)})^2, \quad (5.7)$$

where (4.3*a*) and the definition for $d(n,m)$ given in the appendix has been used. The function $p(A^*)$, which is a function of A^* only, has been plotted in figure 3. We note that it is a monotonically decreasing function.

Since R_1 is positive we find that the action of the Lorentz forces leads to an increase in the Rayleigh number R at a given value of the convection amplitude A. From the physical point of view the Rayleigh number rather than A has to be considered as the prescribed parameter and we have to conclude that the magnetic energy can increase only in conjunction with a decreasing amplitude A according to the relation

$$R = R_\alpha + \tfrac{1}{8}(PA)^2(\pi^2+\alpha^2)^2 + R_1 E. \tag{5.8}$$

The decrease of A caused by an increasing E is accelerated by the fact that R_1 is a monotonically decaying function of A^*. Of course, a growing value of E is only possible as long as A^* exceeds its critical value, which depends on the constant W_0 and is determined by relation (4.10). When A^* reaches the critical value (4.11) in the case of large values of W_0^* or the corresponding higher critical value for moderate values of W_0^* the growth rate vanishes and E attains its equilibrium value E_e. From (5.8) a relation for E_e is readily obtained:

$$E \to E_e = (R - R_\alpha - \tfrac{1}{8}(PA_c^*/P_m)^2\,(\pi^2+\alpha^2)^2)/P_m\, p(A_c^*). \tag{5.9}$$

The fact that the hydrodynamic analysis was based on the condition

$$(R - R_\alpha)/R \ll 1 \tag{5.10}$$

and that P_m was assumed large justifies our anticipation that E is a small parameter. Since R_1 is proportional to P_m the basic balance (5.8) shows that the ratio of the magnetic energy to the gravitational energy released by convection, which is proportional to $(PA)^2$, is of the same order as the ratio between magnetic and viscous diffusivity. Convection with stress-free boundaries is exceptional in that the kinetic energy does not enter the relation (5.8) at order A^2. In the case of rigid boundaries a term proportional to the kinetic energy would become important in the limit of small Prandtl number and the above statement would hold in this case for the kinetic energy in place of the gravitational energy. The fact that E is of order $(PA)^2/P_m$ also justifies the neglect of the other components of the Lorentz force. Since the change of the amplitude of the mean flow is of the order E the relative change of W is small compared with relative change of A as E grows to its equilibrium value. The (z, t)-dependent components of the Lorentz force are even less important for the determination of the equilibrium value E_e. They induce a fluctuating velocity field of the order E which influences the convection amplitude only by its nonlinear interaction, which is of order E^2 and can be neglected for this reason.

In the above analysis the wavenumber β and the details of the dynamo mechanism did not enter. At first sight this fact may appear paradoxical since the work done by the fluid motion in the direction opposite to the Lorentz force must be equal to the ohmic losses of the magnetic field which is generated by the dynamo mechanism. A closer inspection shows, however, that the energy transport can be separated from the dynamo mechanism in the limit of small wavenumbers β.

The action of Lorentz force in the approximation used above is the same as in the case when convection sets in without mean flow in an initially homogeneous magnetic field in the direction perpendicular to the convection rolls. This situation is described by equation (4.3a). Even though no amplification of the basic magnetic field is possible the energy of the magnetic field increases owing to the compression of field lines as shown in figure 2. Associated with this process are ohmic losses, which have to be provided for by the action of the Lorentz force on the convection flow. Relation (5.8) is valid in this case and determines the equilibrium amplitude A of convection as a function of the energy E of the initially given homogeneous field. The assumption of a homogeneous field is, of course, not physical, since any real magnetic field is subject to dissipation, at least in a fluid of finite conductivity. For this reason the dynamo problem requires that all components of the magnetic field possess a characteristic length scale as in the case which has been considered in this paper. Without the presence of the mean flow the average part of the magnetic field decays owing to its z dependence even though the magnetic energy can be increased by the convection flow for a limited time. Only if a dynamo mechanism operates which overcomes the dissipation associated with the z dependence can a stationary state be achieved.

6. Concluding remarks

The model for the generation of magnetic fields by convection which has been analysed in this paper has several idealized features. Though it is possible to generate convection in the presence of stress-free boundaries (Goldstein & Graham 1969) this boundary condition must be regarded as unrealistic as the assumption of infinitely electrically conducting boundaries. It is even less likely that both conditions apply at the same time. Yet, the experience of the theoretical work in the past has shown that virtually all features of convection can be described, at least qualitatively, in the case of free boundaries. The same holds for the infinitely conducting boundary in the case of the magnetic field. Both are natural boundary conditions which allow the continuation of the solution in a periodic fashion. We feel that the distinguished properties of the boundary conditions justify their use even if they can hardly be realized experimentally. We expect that the physically more realistic case of convection rolls with Poiseuille flow in the presence of rigid boundaries will lead qualitatively to the same results.

The dynamo mechanism considered in this paper is related to solutions of the kinematic dynamo problem obtained in earlier work. We have mentioned already the recent paper by G. O. Roberts (1972) on magnetic fields generated by two-dimensional velocity fields. In his terminology the mechanism considered in the present paper is a second-order dynamo. First-order dynamos are characterized, generally speaking, by a finite value of the average helicity, which vanishes for the velocity field derived in §2.

The generation of magnetic fields in the form of waves was first considered by Parker (1955). The details of the mechanism described in §4 differ considerably from Parker's model. The roles of the mean and the fluctuating velocity are, in a sense, reversed. In the case of the earth's magnetic field, which motivated Parker's study, the mean field is thought to be created by the shear of the differential rotation while in the present case the mean shear creates the fluctuating poloidal field. Although our analysis does not apply to the case of the geodynamo the velocity field resembles the motions generated by a convective instability in the earth's core (Busse 1971), which are very nearly two-dimensional because of the dominating constraint of rotation. It is of interest to note at this point that the velocity field (2.6) remains unchanged if the fluid layer is rotating about an axis in the z direction. A modified model, in which the mean flow is directed in the y direction and a fluctuating velocity field in the z direction is induced by Ekman-layer suction, may eventually lead to a dynamo mechanism which is akin to the process generating the earth's magnetic field.

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Appendix

The matrices involving ψ_0 can be written in the form

$$A_{nm\nu\mu} = -d(n,m)\,\delta_{\nu n}\,\delta_{\mu m} - a\sum_{s,r}\delta_{\nu+s,n}\,\delta_{\mu+r,m}(sn-rm)\,(1+\tfrac{1}{2}(r+1)\delta_{m1}),$$

$$B_{nm\nu\mu} = d(\nu,\mu)[d(n,m)\,\delta_{\nu n}\,\delta_{\mu m} + a\sum_{s,r}\delta_{\nu+s,n}\,\delta_{\mu+r,m}(sn-rm)\,(1+\tfrac{1}{2}(s+1)\,\delta_{n1})],$$

$$H_{nm\nu\mu} = \tfrac{1}{4}\sum_{s,r}\delta_{\nu+s,n}\,\delta_{\mu+r,m}[-s(n-s)\,\pi^2 - r(m-r)\,\alpha^2 + d(n-s,m-r)\,(rn\pi^2+sm\alpha^2)/d(n,m)]\,(1+\tfrac{1}{2}(s+1)\,\delta_{m1}),$$

where $d(n,m)$ and a are defined by

$$d(n,m) \equiv n^2\pi^2+m^2\alpha^2, \quad a = \tfrac{1}{4}\alpha\pi A^*.$$

The variables s and r assumes the values $+1$ and -1. The summation is to be extended over all possible combinations. The matrices describing the interaction of the magnetic field with the mean flow are given by

$$E_{nm\nu\mu} = \tfrac{1}{2}\alpha\pi\,\{\sum_s \delta_{\nu+2s,n}\,\delta_{\mu m}(-sm) + \delta_{\nu 1}\,\delta_{n1}\,m\,\delta_{\mu m}\},$$

$$F_{nm\nu\mu} = \frac{1}{2d(n,m)}\{\sum_s \delta_{\nu+2s,n}\,\delta_{\mu m}(n(n-2s)\,\pi^2+m^2\alpha^2) + \delta_{\nu 1}\,\delta_{n1}\,\delta_{\mu m}(m^2\alpha^2-\pi^2)\},$$

$$K_{nm\nu\mu} = -\tfrac{1}{2}\{\sum_s \delta_{\nu+2s,n}\,\delta_{\mu m}(n(n-2s)\,\pi^2+m^2\alpha^2)\,(1+\tfrac{1}{2}(s+1)\,\delta_{n1}) + \delta_{\nu 1}\,\delta_{n1}\,\delta_{\mu m}(m^2\alpha^2-\pi^2)\}.$$

In order to avoid confusion we have inserted in some cases a comma between the subscripts of the unity matrix $\delta_{\nu\mu}$. The variable s assumes again the values ± 1. The sum includes both possibilities.

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